

# Generalizations of $Q$ -systems and Orthogonal Polynomials from Representation Theory

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**ABSTRACT.** We briefly describe what tau-functions in integrable systems are. We then define a collection of tau-functions given as matrix elements for the action of  $\widehat{GL}_2$  on two-component Fermionic Fock space. These tau-functions are solutions to a discrete integrable system called a  $Q$ -system.

We can prove that our tau-functions satisfy  $Q$ -system relations by applying the famous “Desnanot-Jacobi identity” or by using “connection matrices”, the latter of which gives rise to orthogonal polynomials. In this paper, we will provide the background information required for computing these tau-functions and obtaining the connection matrices and will then use the connection matrices to derive our difference relations and to find orthogonal polynomials.

We generalize the above by considering tau-functions that are matrix elements for the action of  $\widehat{GL}_3$  on three-component Fermionic Fock space, and discuss the new system of discrete equations that they satisfy. We will show how to use the connection matrices in this case to obtain “multiple orthogonal polynomials of type II”.

## 1. Introduction

Integrable differential equations, such as the KdV equation,

$$(1.1) \quad u_t + u_{xxx} + 6uu_x = 0,$$

can be solved exactly by employing a change of variables to rewrite the equations more simply in bilinear form. In the case of the KdV equation, this change of variables is given by [6]

$$(1.2) \quad u = 2(\ln \tau)_{xx}.$$

This method of changing variables is called Hirota’s method and the solutions of these differential equations under the change of variables are referred to as “tau-functions” (For more details on using Hirota’s method to find solutions to the KdV equation, as well as many other examples, see [6].)

Interestingly, tau-functions are often equal to matrix elements for representations of infinite dimensional Lie groups (see, for example, [12] and [8]).

In this paper, we will discuss tau-functions that satisfy discrete integrable equations. We will first define tau-functions that are given as matrix elements for the

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action of  $\widehat{GL}_2$  on two-component Fermionic Fock space and will discuss how to show that these tau-functions satisfy a  $Q$ -system.

More specifically, we will see that our  $\widehat{GL}_2$  tau-functions satisfy

$$(1.3) \quad \tau_k^{(\alpha)} \tau_{k-2}^{(\alpha+2)} = \tau_{k-1}^{(\alpha+2)} \tau_{k-1}^{(\alpha)} - (\tau_{k-1}^{(\alpha+1)})^2,$$

for  $k \geq 0$  and  $\alpha \in \mathbb{Z}$ . By applying a suitable change of variables, this can be shown to be equivalent to the defining relations for the  $A_{\infty/2}$   $Q$ -system which is discussed, for example, in [4]. These difference relations are found using “connection matrices” (defined below) and these connection matrices can also be used to obtain orthogonal polynomials.

$Q$ -systems are discrete integrable systems that appear in various places in mathematics, for example, as the relations satisfied by characters of Kirillov-Reshetikhin modules (see [10], [11]) or as mutations in a cluster algebra (see [9], [3]).

Since  $Q$ -systems and orthogonal polynomials are already interesting, it is natural to ask what sort of discrete relations are satisfied by analogous tau-functions, given as matrix elements for the action of  $\widehat{GL}_3$  on three-component Fermionic Fock space and what sort of orthogonal polynomials come from the corresponding connection matrices. In the following, we will describe how to define these new  $\widehat{GL}_3$  tau-functions and how to use connection matrices to show that they satisfy the following system of equations, for all  $k, \ell \geq 0$  and  $\alpha, \beta \in \mathbb{Z}$ ,

$$\begin{aligned} (1) \quad & (\tau_{k,\ell}^{(\alpha+1,\beta)})^2 = \tau_{k,\ell}^{(\alpha,\beta)} \tau_{k,\ell}^{(\alpha+2,\beta)} + \tau_{k+1,\ell+1}^{(\alpha,\beta)} \tau_{k-1,\ell-1}^{(\alpha+2,\beta)} - \tau_{k+1,\ell}^{(\alpha,\beta)} \tau_{k-1,\ell}^{(\alpha+2,\beta)} \\ (2) \quad & \tau_{k,\ell}^{(\alpha+1,\beta)} \tau_{k,\ell-1}^{(\alpha+2,\beta)} = \tau_{k-1,\ell-1}^{(\alpha+2,\beta)} \tau_{k+1,\ell}^{(\alpha+1,\beta)} + \tau_{k,\ell-1}^{(\alpha+1,\beta)} \tau_{k,\ell}^{(\alpha+2,\beta)} \\ (3) \quad & (\tau_{k,\ell}^{(\alpha,\beta+1)})^2 = \tau_{k,\ell}^{(\alpha,\beta)} \tau_{k,\ell}^{(\alpha,\beta+2)} - \tau_{k,\ell-1}^{(\alpha,\beta+2)} \tau_{k,\ell+1}^{(\alpha,\beta)} - \tau_{k+1,\ell}^{(\alpha,\beta+2)} \tau_{k-1,\ell}^{(\alpha,\beta)} \\ (4) \quad & \tau_{k-1,\ell}^{(\alpha,\beta+1)} \tau_{k,\ell+1}^{(\alpha,\beta)} = \tau_{k-1,\ell}^{(\alpha,\beta)} \tau_{k,\ell+1}^{(\alpha,\beta+1)} + \tau_{k-1,\ell+1}^{(\alpha,\beta)} \tau_{k,\ell}^{(\alpha,\beta+1)}. \end{aligned}$$

We hope, similarly to  $Q$ -systems, that our new system of equations will also have connections to other areas of mathematics. We will briefly discuss progress we have made in analyzing this new system of equations.

Applying restrictions to the connection matrices in the  $\widehat{GL}_3$  case, we find an analogous collection of orthogonal polynomials, which we will discuss. In our future work, we hope to investigate more general situations, obtained by dropping these restrictions.

(See [1] for more details on the computations of our tau-functions and the difference relations that they satisfy. Orthogonal polynomials, however are not discussed there.)

## 2. Calculating $\widehat{GL}_2$ Tau-Functions on Two-Component Fermionic Fock Space

Before we define our  $\widehat{GL}_2$  tau-functions, we first define two-component Fermionic Fock space,  $F^{(2)}$  and describe the action of  $\widehat{GL}_2$  on this space. Here, we will omit most technical details. For more information, we refer the reader to [15] and [1]. In particular, all omitted details of the following background on Fermionic Fock space and the associated action of  $\widehat{gl}_2$  can be found in [1].

Consider the vector space  $H^{(2)} := \mathbb{C}^2 \otimes \mathbb{C}[z, z^{-1}]$ . A basis of this space is given by elements,  $e_a z^k$ ,  $a = 0, 1$ ,  $k \in \mathbb{Z}$ , where  $e_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .  $F^{(2)}$  is then

spanned by vectors,

$$w = w_0 \wedge w_1 \wedge w_2 \wedge \cdots,$$

where  $w_i \in H^{(2)}$  and the  $w$  satisfy some restrictions that we will now discuss.

Let the vacuum vector be

$$v_0 := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} z \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ z \end{pmatrix} \wedge \cdots \in F^{(2)},$$

and define operators,  $e(e_a z^k)$  and  $i(e_a z^k)$  (called exterior and interior product operators, respectively), given by  $e(e_a z^k)w = e_a z^k \wedge w$  and  $i(e_a z^k)w = \beta$  if  $w = e_a z^k \wedge \beta$ .  $F^{(2)}$  is the span of the vectors obtained by acting on  $v_0$  by finitely many exterior and interior product operators. We can specify an order in which to act by these exterior and interior product operators and define “elementary wedges” as those wedges obtained by acting on  $v_0$  by monomials of exterior and interior product operators, subject to this order. For more information, see [15] and [1]. The elementary wedges are defined in such a way that there exists a unique bilinear form on  $F^{(2)}$ , denoted  $\langle v, w \rangle$  for  $v, w \in F^{(2)}$ , for which the elementary wedges are an orthonormal basis.

It is useful to introduce generating series, called fermion fields, for the exterior and interior product operators. Define

$$\psi_a^\pm(w) = \sum_{k \in \mathbb{Z}} {}_a \psi_{(k)}^\pm w^{-k-1}, \quad a = 0, 1,$$

where

$${}_a \psi_{(k)}^+ = e(e_a z^k) \text{ and } {}_a \psi_{(k)}^- = i(e_a z^{-k-1}).$$

We can use fermion fields to express the action of  $\widehat{gl}_2 = gl_2 \otimes \mathbb{C}[z, z^{-1}] \oplus \mathbb{C}c$  on  $F^{(2)}$ . Let  $E_{ab} \in gl_2$  ( $a, b = 0, 1$ ) be the matrices such that  $E_{ab}e_c = \delta_{bc}e_a$  and let the current,  $E_{ab}(w) = \sum_{k \in \mathbb{Z}} E_{ab} z^k w^{-k-1}$ , be the generating series of elements in  $\widehat{gl}_2$ .

When  $a \neq b$ , the series acts on  $F^{(2)}$  by

$$(2.1) \quad E_{ab}(w) = \psi_a^+(w) \psi_b^-(w).$$

The action of  $E_{ab}(w)$  in general requires using the normal ordered product, but we do not discuss this here since it is not needed in this paper (More details can be found in [1].).

In addition to the action of the Lie algebra,  $\widehat{gl}_2$ ,  $F^{(2)}$  also carries an action of the group  $\widehat{GL}_2$ , a central extension of the loop group,  $\widetilde{GL}_2$ . In particular, on  $F^{(2)}$  we have the action of “fermionic translation operators”  $Q_0, Q_1$  such that

$$\pi(Q_0) = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \quad \pi(Q_1) = \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}$$

where  $\pi$  denotes the projection from  $\widehat{GL}_2$  to  $\widetilde{GL}_2$ . We also define  $T = Q_1 Q_0^{-1}$  such that

$$\pi(T) = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}.$$

Let  $g_C \in \widehat{GL}_2$  be such that

$$\pi(g_C) = \begin{bmatrix} 1 & 0 \\ C(z) & 1 \end{bmatrix},$$

where  $C(z) = \sum_{i \in \mathbb{Z}} \frac{c_i}{z^{i+1}}$ ,  $c_i \in \mathbb{C}$ . (Actually, for our purposes, it will sometimes be useful to take the  $c_i$ s to be formal variables.)

We define our (“unshifted”) tau-functions to be

$$\tau_k = \langle T^k v_0, g_C \cdot v_0 \rangle.$$

We also need to define “shifted” tau-functions, corresponding to the action of  $g_C^{(\alpha)}$  on the vacuum vector, where

$$g_C^{(\alpha)} = Q_0^\alpha g_C Q_0^{-\alpha}, \quad \alpha \in \mathbb{Z}.$$

Denote by  $C^{(\alpha)}$ , the series  $C^{(\alpha)}(z) = \sum_{i \in \mathbb{Z}} \frac{c_{i+\alpha}}{z^{i+1}}$ , so that  $\pi(g_C^{(\alpha)}) = \begin{bmatrix} 1 & 0 \\ C^{(\alpha)}(z) & 1 \end{bmatrix}$ .

We can calculate these tau-functions by noting that

$$g_C^{(\alpha)} = \exp \begin{bmatrix} 0 & 0 \\ C^{(\alpha)}(z) & 0 \end{bmatrix} = \exp(\text{Res}_w(C^{(\alpha)}(w)E_{10}(w))),$$

where the action of the current,  $E_{10}(w)$  on  $F^{(2)}$ , is given by (2.1). We then have the following formulas for our tau-functions, which are stated and proven in [1]:

- THEOREM 2.1.** (1)  $\tau_k^{(\alpha)} = 0$  for  $k < 0$ .  
 (2)  $\tau_0^{(\alpha)} = 1$ .  
 (3) When  $k > 0$ ,

$$\tau_k^{(\alpha)} = \frac{1}{k!} \text{Res}_{\mathbf{w}} \left( \prod_{1 \leq i < j \leq k} (w_i - w_j)^2 \prod_{i=1}^k C^{(\alpha)}(w_i) \right),$$

where  $\text{Res}_{\mathbf{w}}$  denotes  $\text{Res}_{w_1}(\text{Res}_{w_2} \cdots (\text{Res}_{w_k} \cdots) \cdots)$ .

Alternatively, when  $k > 0$  we can write

$$\tau_k^{(\alpha)} = \det \begin{bmatrix} c_\alpha & c_{\alpha+1} & \cdots & c_{\alpha+k-1} \\ c_{\alpha+1} & c_{\alpha+2} & \cdots & c_{\alpha+k} \\ \vdots & \vdots & \cdots & \vdots \\ c_{\alpha+k-1} & c_{\alpha+k} & \cdots & c_{\alpha+2k-2} \end{bmatrix}.$$

We note in particular that these tau-functions are determinants of Hankel matrices and are thus especially well suited to applying the famous Desnanot-Jacobi identity [2]. We obtain

$$(2.2) \quad \tau_k^{(\alpha)} \tau_{k-2}^{(\alpha+2)} = \tau_{k-1}^{(\alpha+2)} \tau_{k-1}^{(\alpha)} - (\tau_{k-1}^{(\alpha+1)})^2,$$

for  $k \geq 0$  and  $\alpha \in \mathbb{Z}$ .

### 3. Generalizing to $\widehat{GL}_3$

We now generalize the above to the  $\widehat{GL}_3$  case. Here, in the same way that we did in the  $\widehat{GL}_2$  case, we first define the action of  $\widehat{gl}_3$  on three-component Fermionic Fock space,  $F^{(3)}$ . We define  $H^{(3)} := \mathbb{C}^3 \otimes \mathbb{C}[z, z^{-1}]$ , which has a basis given by

$$e_a z^k, \text{ where } a = 0, 1, 2, k \in \mathbb{Z}, \text{ and } e_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The vacuum vector is

$$v_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} z \\ 0 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ z \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \wedge \cdots \in F^{(3)}.$$

Similarly to the  $\widehat{GL}_2$  case, in addition to the action of  $\widehat{gl}_3$  on  $F^{(3)}$ , we have an associated action of  $\widehat{GL}_3$ , a central extension of the loop group,  $\widehat{GL}_3$ . We have as before, a projection map,  $\pi$  from  $\widehat{GL}_3$  to  $\widehat{gl}_3$ . We define fermionic translation operators,  $Q_0, Q_1$ , and  $Q_2$  such that

$$\pi(Q_0) = \begin{bmatrix} z^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \pi(Q_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \pi(Q_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}.$$

We also need operators,  $T_1 = Q_1 Q_0^{-1}$  and  $T_2 = Q_2 Q_1^{-1}$  such that

$$\pi(T_1) = \begin{bmatrix} z & 0 & 0 \\ 0 & z^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \pi(T_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}.$$

The fermion fields and action of  $\widehat{gl}_3$  are defined in a way completely analogous to how they were defined in our discussion of the  $\widehat{GL}_2$  case. As before, we omit any discussion of the normal ordered product, since the details are not necessary for this paper and can be found in [1].

We take a loop group element,  $g_{C,D,E} \in \widehat{GL}_3$ , such that

$$\pi(g_{C,D,E}) = \begin{bmatrix} 1 & 0 & 0 \\ C(z) & 1 & 0 \\ D(z) & E(z) & 1 \end{bmatrix},$$

where  $C(z) = \sum_{i \in \mathbb{Z}} \frac{c_i}{z^{i+1}}$ ,  $D(z) = \sum_{i \in \mathbb{Z}} \frac{d_i}{z^{i+1}}$ , and  $E(z) = \sum_{i \in \mathbb{Z}} \frac{e_i}{z^{i+1}}$  and the  $c_i$ s,  $d_i$ s, and  $e_i$ s are complex numbers or formal variables.

We define

$$\tau_{k,\ell} = \langle g_{C,D,E} \cdot v_0, T_1^k T_2^\ell v_0 \rangle,$$

and calculate the action of  $g_{C,D,E}$  in the same way that we calculated the action of our previous group element on the two-component Fermionic Fock space, by expressing  $g_{C,D,E}$  in terms of fermion fields.

In order to obtain our difference relations we must, as in the  $\widehat{GL}_2$  case, introduce shifted tau-functions. Here, we have two independent shifts. We define the  $\tau_{k,\ell}^{(\alpha,\beta)}$ s to be the tau-functions corresponding to the action of the group element,

$$g_{C,D,E}^{(\alpha,\beta)} = Q_0^{-\alpha} Q_1^{-\beta} g_{C,D,E} Q_1^\beta Q_0^\alpha.$$

We comment that we do not need  $Q_2$  to obtain all possible shifts since

$$Q_2 g_{C,D,E} Q_2^{-1} = Q_0^{-1} Q_1^{-1} g_{C,D,E} Q_1 Q_0,$$

so there really are only two independent shifts, as claimed.

The formula for our  $\tau_{k,\ell}^{(\alpha,\beta)}$  functions is then: (The following is stated and proven in [1]).

THEOREM 3.1.

$$\tau_{k,\ell}^{(\alpha,\beta)} = \sum_{n_c+n_d=k, n_e+n_d=\ell} c_{n_c, n_d, n_e}$$

where

$$c_{n_c, n_d, n_e} = \frac{1}{n_c! n_d! n_e!} \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{y}} \text{Res}_{\mathbf{z}} \left( \prod_{i=1}^{n_c} C^{(\alpha-\beta)}(x_i) \prod_{i=1}^{n_d} D^{(\alpha)}(y_i) \prod_{i=1}^{n_e} E^{(\beta)}(z_i) p_{n_c, n_d, n_e} \right)$$

where we use the same notation for residues as we did in the  $\widehat{GL}_2$  case and

$$p_{n_c, n_d, n_e} = (-1)^{\frac{n_d(n_d+1)}{2}} \times \frac{\prod_{1 \leq i < j \leq n_c} (x_i - x_j)^2 \prod_{1 \leq i < j \leq n_d} (y_i - y_j)^2 \prod_{1 \leq i < j \leq n_e} (z_i - z_j)^2 \prod_{i=1}^{n_c} \prod_{j=1}^{n_d} (x_i - y_j) \prod_{i=1}^{n_d} \prod_{j=1}^{n_e} (y_i - z_j)}{\prod_{i=1}^{n_c} \prod_{j=1}^{n_e} (x_i - \underline{z_j})}.$$

Here, the underline means that we expand  $\frac{1}{x_i - z_j}$  in positive powers of  $z_j$ . We comment that the formula given here is analogous to the formula for our  $\widehat{GL}_2$  tau-functions, but is much more complicated. In particular, the denominator appearing in our formula here means that our tau-functions are, in general, infinite series of monomials in the  $c_i$ s,  $d_i$ s, and  $e_i$ s.

#### 4. Connection Matrices and Zero Curvature Equations

Although the Desnanot-Jacobi identity easily yields difference relations for our  $\widehat{GL}_2$  tau-functions, it is unclear how one might use it to find difference relations for our  $\widehat{GL}_3$  tau-functions. To find difference relations for our  $\widehat{GL}_3$  tau-functions, we use “connection matrices” and show that certain zero curvature equations are satisfied. For this, we need the Birkhoff factorization of  $T_2^{-\ell} T_1^{-k} g_{C,D,E}^{(\alpha,\beta)}$ , which exists when  $\tau_{k,\ell}^{(\alpha,\beta)} \neq 0$  ([13]). Here, we consider our  $c_i$ s,  $d_i$ s, and  $e_i$ s to be formal variables and assume that the Birkhoff factorization exists for all  $T_2^{-\ell} T_1^{-k} g_{C,D,E}^{(\alpha,\beta)}$  where  $k$  and  $\ell$  are both nonnegative and  $\alpha, \beta \in \mathbb{Z}$ .

**4.1.  $\widehat{GL}_2$  Case.** In this section, we will use zero curvature equations ([5]) to rederive our difference relations for the  $\widehat{GL}_2$  case. The  $\widehat{GL}_3$  case is similar, the details of which are included in [1].

For the  $\widehat{GL}_2$  case, we have a factorization

$$T^{-k} g_C^{(\alpha)} = g_-^{[k](\alpha)} g_+^{[k](\alpha)},$$

where  $g_-^{[k](\alpha)}$  is such that  $\pi(g_-^{[k](\alpha)}) = I +$  terms involving only negative powers of  $z$  and  $g_+^{[k](\alpha)}$  is such that  $\pi(g_+^{[k](\alpha)}) = A_k^{(\alpha)} +$  terms involving only positive powers of  $z$ . Here,  $I$  denotes the identity matrix and  $A_k^{(\alpha)}$  denotes a matrix that is independent of both  $z$  and  $z^{-1}$ . This factorization of  $T^{-k} g_C^{(\alpha)}$  is called the Birkhoff factorization ([13]). From now on, we will work in the non-centrally extended loop group, i.e. in  $\widehat{GL}_2$ , but omit  $\pi$  in our notation.

We define matrix Baker functions,  $\Psi^{[k](\alpha)}$  by

$$\Psi^{[k](\alpha)} = T^k g_-^{[k](\alpha)}.$$

We then have connection matrices,  $U_k^{(\alpha)}$ , between these matrix Baker functions, given by

$$U_k^{(\alpha)} = (\Psi^{[k](\alpha)})^{-1} \Psi^{[k+1](\alpha)}.$$

The entries of these connection matrices have expressions in terms of the tau-functions. (Another fact about the connections matrices, which will be particularly important in our discussion of orthogonal polynomials in the next section, is that they are nonnegative in  $z$ , i.e. none of their entries have negative powers of  $z$ . [1])

We obtain the difference relations by factoring our connection matrices,

$$U_k^{(\alpha)} = V_k^{(\alpha)} (W_k^{(\alpha)})^{-1} = (W_k^{(\alpha-1)})^{-1} V_{k+1}^{(\alpha-1)},$$

where

$$V_k^{(\alpha)} = (g_-^{[k](\alpha)})^{-1} Q_0^{-1} g_-^{[k](\alpha+1)}$$

and

$$W_k^{(\alpha)} = (g_-^{[k+1](\alpha)})^{-1} Q_1^{-1} g_-^{[k](\alpha+1)}.$$

The equality of these two different expressions for the connection matrices are called “zero curvature equations” and they give us equalities satisfied by the tau-functions. These equalities imply the difference relations previously found using the Desnanot-Jacobi identity.

More explicitly, we have the following lemma, the proof of which is included in [1]:

LEMMA 4.1. [1]

$$V_k^{(\alpha)} = \begin{bmatrix} z - \frac{\tau_{k-1}^{(\alpha+1)} \tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha+1)} \tau_k^{(\alpha)}} & \frac{\tau_{k-1}^{(\alpha+1)}}{\tau_k^{(\alpha+1)}} \\ -\frac{\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha)}} & 1 \end{bmatrix}, \quad W_k^{(\alpha)} = \begin{bmatrix} 1 & -\frac{\tau_k^{(\alpha)}}{\tau_{k+1}^{(\alpha)}} \\ \frac{\tau_{k+1}^{(\alpha+1)}}{\tau_k^{(\alpha+1)}} & z - \frac{\tau_k^{(\alpha)} \tau_{k+1}^{(\alpha+1)}}{\tau_{k+1}^{(\alpha)} \tau_k^{(\alpha+1)}} \end{bmatrix}.$$

Since  $V_k^{(\alpha)}$  and  $W_k^{(\alpha)}$  satisfy,

$$U_k^{(\alpha)} = V_k^{(\alpha)} (W_k^{(\alpha)})^{-1} = (W_k^{(\alpha-1)})^{-1} V_{k+1}^{(\alpha-1)},$$

we find that

$$U_k^{(\alpha)} = \begin{bmatrix} z - \frac{\tau_k^{(\alpha)} \tau_{k+1}^{(\alpha+1)}}{\tau_{k+1}^{(\alpha)} \tau_k^{(\alpha+1)}} - \frac{\tau_{k-1}^{(\alpha+1)} \tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha+1)} \tau_k^{(\alpha)}} & \frac{\tau_k^{(\alpha)}}{\tau_{k+1}^{(\alpha)}} \\ -\frac{\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha)}} & 0 \end{bmatrix} = \begin{bmatrix} z - \frac{\tau_k^{(\alpha)} \tau_{k+2}^{(\alpha-1)}}{\tau_{k+1}^{(\alpha)} \tau_{k+1}^{(\alpha-1)}} - \frac{\tau_k^{(\alpha-1)} \tau_{k+1}^{(\alpha)}}{\tau_{k+1}^{(\alpha-1)} \tau_k^{(\alpha)}} & \frac{\tau_k^{(\alpha)}}{\tau_{k+1}^{(\alpha)}} \\ -\frac{\tau_{k+1}^{(\alpha)}}{\tau_k^{(\alpha)}} & 0 \end{bmatrix}.$$

This implies

$$(\tau_k^{(\alpha)})^2 (\tau_{k+2}^{(\alpha-1)} \tau_k^{(\alpha+1)} - \tau_{k+1}^{(\alpha-1)} \tau_{k+1}^{(\alpha+1)}) = (\tau_{k+1}^{(\alpha)})^2 (\tau_{k+1}^{(\alpha-1)} \tau_{k-1}^{(\alpha+1)} - \tau_k^{(\alpha-1)} \tau_k^{(\alpha+1)}).$$

We notice that, if

$$\tau_{k+1}^{(\alpha-1)} \tau_{k-1}^{(\alpha+1)} = \tau_k^{(\alpha+1)} \tau_k^{(\alpha-1)} - (\tau_k^{(\alpha)})^2$$

holds for some  $k$ , the above identity implies that it holds for  $k + 1$ . Since  $\tau_{-1}^{(\alpha)} = 0$  and  $\tau_0^{\alpha-1} = 1$ ,  $\tau_{k+1}^{(\alpha-1)}\tau_{k-1}^{(\alpha+1)} = \tau_k^{(\alpha+1)}\tau_k^{(\alpha-1)} - (\tau_k^{(\alpha)})^2$  holds trivially for  $k = 0$ , and hence

$$\tau_{k+1}^{(\alpha-1)}\tau_{k-1}^{(\alpha+1)} = \tau_k^{(\alpha+1)}\tau_k^{(\alpha-1)} - (\tau_k^{(\alpha)})^2$$

holds for all  $k \geq 0$ , which is exactly what we previously obtained using the Desnanot-Jacobi identity (1.3).

**4.2.  $\widehat{GL}_3$  Case.** The matrix Baker functions in the  $\widehat{GL}_3$  case are given in a completely analogous way to the  $\widehat{GL}_2$  case. Here,

$$\Psi^{[k,\ell](\alpha,\beta)} = T_1^k T_2^\ell g_-^{[k,\ell](\alpha,\beta)},$$

where  $g_-^{[k,\ell](\alpha,\beta)}$  is the part of the Birkhoff factorization of  $T_2^{-\ell} T_1^{-k} g_{C,D,E}^{(\alpha,\beta)}$  that is a lift of  $I$  plus a  $3 \times 3$  matrix whose entries have only negative powers of  $z$ . (As before,  $I$  denotes the identity matrix.)

We then have two sets of connection matrices,  $U_{[k+,\ell]}^{(\alpha,\beta)}$  and  $U_{[k,\ell+]}^{(\alpha,\beta)}$ , that we factor in two different ways to obtain our zero curvature equations.

$$\begin{aligned} U_{[k+,\ell]}^{(\alpha,\beta)} &= (\Psi^{[k,\ell](\alpha,\beta)})^{-1} \Psi^{[k+1,\ell](\alpha,\beta)}, \\ U_{[k,\ell+]}^{(\alpha,\beta)} &= (\Psi^{[k,\ell](\alpha,\beta)})^{-1} \Psi^{[k,\ell+1](\alpha,\beta)}. \end{aligned}$$

Factoring these connection matrices and performing calculations similar to those above, we find that our  $\widehat{GL}_3$  tau-functions satisfy the system of four difference equations listed in the introduction. Under suitable changes of variables, equations (2) and (4) can be shown to be  $T$ -system relations.  $T$ -systems are a sort of generalization of  $Q$ -systems which have interpretations, for example in terms of perfect matchings of graphs ([14]). Like  $Q$ -system relations,  $T$ -system relations can also be seen as cluster algebra mutations ([3]). We comment that equations (2) and (4) are independent of each other ((2) depends only on the  $k, \ell, \alpha$  parameters and equation (4) depends on the  $k, \ell, \beta$  parameters), so our tau-functions satisfy  $T$ -system relations in two different ways. We believe that the remaining two relations, (1) and (3), are independent of each other and are not implied by the  $T$ -system relations. We are currently working to find other situations in which equations (1) and (3) appear.

## 5. Orthogonal Polynomials from Connection Matrices

**5.1.  $\widehat{GL}_2$  Case.** In the  $\widehat{GL}_2$  case, we can use the fact that the connection matrices are nonnegative in  $z$  to obtain orthogonal polynomials. In the following, we will explain how to do this. (For more on orthogonal polynomials see, for example, [7]).

**DEFINITION 5.1.** [1] Define  $S : \mathbb{C}[c_k]_{k \in \mathbb{Z}} \rightarrow \mathbb{C}[c_k]_{k \in \mathbb{Z}}$  to be the multiplicative map such that  $S(1) = 0$  and  $S(c_k) = c_{k+1}$  for all  $k$ . The shift fields,  $S^\pm(z)$ , are then given by  $S^\pm(z) = (1 - \frac{S}{z})^\pm$ , which also act multiplicatively.

**EXAMPLE 5.2.**

$$\begin{aligned} S^+(z)\tau_2^{(\alpha)} &= S^+(z) \det \begin{bmatrix} c_\alpha & c_{\alpha+1} \\ c_{\alpha+1} & c_{\alpha+2} \end{bmatrix} = (c_\alpha - c_{\alpha+1}/z)(c_{\alpha+2} - c_{\alpha+3}/z) - (c_{\alpha+1} - c_{\alpha+2}/z)^2 = \\ &= \det \begin{bmatrix} c_\alpha & c_{\alpha+1} \\ c_{\alpha+1} & c_{\alpha+2} \end{bmatrix} - \det \begin{bmatrix} c_\alpha & c_{\alpha+1} \\ c_{\alpha+2} & c_{\alpha+3} \end{bmatrix} / z + \det \begin{bmatrix} c_{\alpha+1} & c_{\alpha+2} \\ c_{\alpha+2} & c_{\alpha+3} \end{bmatrix} / z^2 = \end{aligned}$$



$$= \tau_2^{(\alpha)} - \det \begin{bmatrix} c_\alpha & c_{\alpha+1} \\ c_{\alpha+2} & c_{\alpha+3} \end{bmatrix} / z + \tau_2^{(\alpha+1)} / z^2$$

We note that the positive shift field,  $S^+(z)$  sends elements of  $\mathbb{C}[c_k]_{k \in \mathbb{Z}}$  to polynomials in  $z^{-1}$  with coefficients in  $\mathbb{C}[c_k]_{k \in \mathbb{Z}}$ . In particular, as is illustrated in the above example,  $S^+(z)$  sends  $\tau_k^{(\alpha)}$  to a Laurent polynomial in  $z$ , with smallest degree equal to  $-k$  and largest degree equal to 0. We also have the following useful formula:

$$(5.1) \quad z^k S^+(z) \tau_k^{(\alpha)} = \det \begin{bmatrix} c_\alpha & c_{\alpha+1} & \cdots & c_{\alpha+k-1} & 1 \\ c_{\alpha+1} & c_{\alpha+2} & \cdots & c_{\alpha+k} & z \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ c_{\alpha+k} & c_{\alpha+k+1} & \cdots & c_{\alpha+2k-1} & z^k \end{bmatrix}$$

for all  $\alpha \in \mathbb{Z}$  and for all  $k \geq 0$ . (This comes from the fact that

$$z^k S^+(z) \tau_k^{(\alpha)} = \frac{1}{k!} \text{Res}_{\mathbf{w}} \left( \prod_{1 \leq i < j \leq k} (w_i - w_j)^2 \prod_{i=1}^k (z - w_i) \prod_{i=1}^k C^{(\alpha)}(w_i) \right).$$

Observe that formulas like (5.1) appear in the theory of orthogonal polynomials. (See for example, equation (2.1.6) in [7]). Below, we will use connection matrices to derive the orthogonality of the polynomials given by (5.1).

The negative shift field,  $S^-(z)$ , sends elements of  $\mathbb{C}[c_k]_{k \in \mathbb{Z}}$  to series in  $z^{-1}$  with coefficients in  $\mathbb{C}[c_k]_{k \in \mathbb{Z}}$ . For example,  $S^-(z) c_\alpha = \sum_{i=0}^{\infty} \frac{c_{\alpha+i}}{z^i}$ .

$$\text{THEOREM 5.3. [1]} \quad \pi(g_-^{[k],(\alpha)}) = \frac{1}{\tau_k^{(\alpha)}} \begin{bmatrix} S^+(z) \tau_k^{(\alpha)} & S^+(z) \tau_{k-1}^{(\alpha)} / z \\ S^-(z) \tau_{k+1}^{(\alpha)} / z & S^-(z) \tau_k^{(\alpha)} \end{bmatrix}$$

Recall the definition of the Baker functions,  $\Psi^{[k](\alpha)} = T^k g_-^{[k](\alpha)}$ . Any two Baker functions are related by elementary connection matrices. In particular,

$$(\Psi^{[0](\alpha)})^{-1} \Psi^{[k](\alpha)} = U_0^{(\alpha)} U_1^{(\alpha)} \cdots U_{k-1}^{(\alpha)}$$

is a product of connection matrices and so is nonnegative in  $z$ .

Using the above theorem, we see that

$$(\Psi^{[0](\alpha)})^{-1} \Psi^{[k](\alpha)} = \begin{bmatrix} 1 & 0 \\ -\sum_{i=0}^{\infty} \frac{c_{\alpha+i}}{z^{i+1}} & 1 \end{bmatrix} \frac{1}{\tau_k^{(\alpha)}} \begin{bmatrix} z^k S^+(z) \tau_k^{(\alpha)} & z^{k-1} S^+(z) \tau_{k-1}^{(\alpha)} \\ z^{-k-1} S^-(z) \tau_{k+1}^{(\alpha)} & z^{-k} S^-(z) \tau_k^{(\alpha)} \end{bmatrix}.$$

Given  $\alpha$ , denote by  $\langle \cdot, \cdot \rangle$  the bilinear product given by

$$(5.2) \quad \langle f(z), g(z) \rangle = \text{Res}_z \left( \sum_{i=0}^{\infty} \frac{c_{\alpha+i}}{z^{i+1}} f(z) g(z) \right),$$

for all polynomials  $f(z)$  and  $g(z)$ , and denote

$$(5.3) \quad p_k^{(\alpha)}(z) = \frac{1}{\tau_k^{(\alpha)}} z^k S^+(z) \tau_k^{(\alpha)},$$

which we note is a monic polynomial of degree  $k$ . Consider the entry in the first column and second row of  $(\Psi^{[0]}(\alpha))^{-1}\Psi^{[k]}(\alpha)$ . Since  $(\Psi^{[0]}(\alpha))^{-1}\Psi^{[k]}(\alpha)$  is nonnegative in  $z$  and  $z^{-k-1}S^-(z)\tau_{k+1}^{(\alpha)}$  has highest degree equal to  $-k-1$ , we see that

$$\langle p_k^{(\alpha)}(z), z^n \rangle = 0$$

for all  $0 \leq n < k$ . So

$$\langle p_k^{(\alpha)}(z), p_\ell^{(\alpha)}(z) \rangle = 0$$

for  $k \neq \ell$  and the  $\{p_k^{(\alpha)}\}$  are a collection of orthogonal polynomials. Since the series  $C^{(\alpha)}(z)$  can be defined arbitrarily, we in fact obtain all orthogonal polynomials in this way. So the theory of orthogonal polynomials appears as a subset in the study of the representation theory of  $\widehat{GL}_2$ .

EXAMPLE 5.4. Hermite polynomials are the monic polynomials orthogonal for the following bilinear form:

$$(5.4) \quad \langle f(z), g(z) \rangle = \int_{-\infty}^{\infty} f(z)g(z)e^{z^2} dz,$$

where the integral is along the real axis. The moments for this bilinear form are given by

$$(5.5) \quad c_i = \int_{-\infty}^{\infty} z^i e^{-z^2} dz = \begin{cases} 0 & i \text{ is odd} \\ \frac{(2m)! \sqrt{\pi}}{m! 2^m} & i = 2m. \end{cases}$$

We can then rewrite (5.4) as

$$(5.6) \quad \langle f(z), g(z) \rangle = \text{Res}_z \left( \sum_{i=0}^{\infty} \frac{c_i}{z^{i+1}} f(z)g(z) \right),$$

which is precisely (5.2) for our specified moments,  $c_i$ ,  $i \geq 0$ , for  $\alpha = 0$ . So, in this case, the orthogonal polynomials given by (5.3) are precisely the Hermite polynomials (see, [7]).

**5.2.  $\widehat{GL}_3$  Case.** We obtain orthogonal polynomials from the  $\widehat{GL}_3$  case exactly as we did in the  $\widehat{GL}_2$  case, by using the connection matrices. For this, we need the analogue of Theorem 5.3 which is as follows. (This theorem is stated and proven in [1].):

THEOREM 5.5.  $\pi(g_-^{[k,\ell]}(\alpha,\beta)) =$

$$\frac{1}{\tau_{k,\ell}^{(\alpha,\beta)}} \begin{bmatrix} S_c^+(z)S_d^+(z)\tau_{k,\ell}^{(\alpha,\beta)} & \frac{S_c^+(z)S_d^+(z)\tau_{k-1,\ell}^{(\alpha,\beta)}}{z} & (-1)^k \frac{S_c^+(z)S_d^+(z)\tau_{k-1,\ell-1}^{(\alpha,\beta)}}{z} \\ \frac{S_c^-(z)S_e^+(z)\tau_{k+1,\ell}^{(\alpha,\beta)}}{z} & S_c^-(z)S_e^+(z)\tau_{k,\ell}^{(\alpha,\beta)} & \frac{S_c^-(z)S_e^+(z)\tau_{k,\ell-1}^{(\alpha,\beta)}}{z} \\ (-1)^{k+1} \frac{S_d^-(z)S_e^-(z)\tau_{k+1,\ell+1}^{(\alpha,\beta)}}{z} & (-1)^k \frac{S_d^-(z)S_e^-(z)\tau_{k,\ell+1}^{(\alpha,\beta)}}{z} & S_d^-(z)S_e^-(z)\tau_{k,\ell}^{(\alpha,\beta)} \end{bmatrix}$$

Here, the shift operators,  $S_c^\pm(z)$ ,  $S_d^\pm(z)$ , and  $S_e^\pm(z)$  are analogous to the shift operators defined in the  $\widehat{GL}_2$  case.  $S_c^\pm(z)$  acts on the  $c_i$ s exactly as  $S^\pm(z)$  does in the  $\widehat{GL}_2$  case and acts trivially on the  $d_i$ s and  $e_i$ s.  $S_d^\pm(z)$  and  $S_e^\pm(z)$  are similarly defined.

The matrix Baker functions for the  $\widehat{GL}_3$  case are given above, by  $\Psi^{[k,\ell]}(\alpha,\beta) = T_1^k T_2^\ell g_-^{[k,\ell]}(\alpha,\beta)$ . We then see that

$$(\Psi^{[0,0]}(\alpha,\beta))^{-1}\Psi^{[k,\ell]}(\alpha,\beta)$$

is nonnegative in  $z$ , since it is equal to a product of connection matrices:

$$(\Psi^{[0,0]}(\alpha, \beta))^{-1} \Psi^{[k,\ell]}(\alpha, \beta) = U_{[0+,0]}^{(\alpha, \beta)} U_{[1+,0]}^{(\alpha, \beta)} \cdots U_{[k-1+,0]}^{(\alpha, \beta)} U_{[k,0+]}^{(\alpha, \beta)} U_{[k,1+]}^{(\alpha, \beta)} \cdots U_{[k,\ell-1+]}^{(\alpha, \beta)}.$$

In the  $\widehat{GL}_2$  case, our orthogonal polynomials were obtained by acting on tau-functions by the shift operators. Since our  $\widehat{GL}_3$  tau-functions are, in general, infinite sums, to obtain polynomials we restrict to the case that either the series  $C^{(\alpha-\beta)}(z)$  or the series  $E^{(\beta)}(z)$  is 0. Here, we discuss the case that  $E^{(\beta)}(z) = 0$ .

Using the formula (in terms of residues) for the  $\widehat{GL}_3$  tau-functions given earlier, we can show that when  $E^{(\beta)}(z) = 0$ ,

$$\tau_{k,\ell}^{(\alpha, \beta)} = 0 \text{ when } k < \ell$$

and when  $k \geq \ell$ ,

$$\tau_{k,\ell}^{(\alpha, \beta)} = (-1)^{\frac{\ell(\ell+1)}{2}} \det \begin{bmatrix} d_\alpha & \cdots & d_{\alpha+\ell-1} & c_{\alpha-\beta} & \cdots & c_{\alpha-\beta+k-\ell-1} \\ d_{\alpha+1} & \cdots & d_{\alpha+\ell} & c_{\alpha-\beta+1} & \cdots & c_{\alpha-\beta+k-\ell} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ d_{\alpha+k-1} & \cdots & d_{\alpha+k-\ell-2} & c_{\alpha-\beta+k-1} & \cdots & c_{\alpha-\beta+2k-\ell-2} \end{bmatrix}.$$

To see this, note that when  $E^{(\beta)}(z) = 0$ ,  $p_{n_c, n_d, n_e}$  in the formula for our  $\tau_{k,\ell}^{(\alpha, \beta)}$  reduces to

$$p_{n_c, n_d} = (-1)^{\frac{n_d(n_d+1)}{2}} \times \prod_{1 \leq i < j \leq n_c} (x_i - x_j)^2 \prod_{1 \leq i < j \leq n_d} (y_i - y_j)^2 \prod_{i=1}^{n_c} \prod_{j=1}^{n_d} (x_i - y_j),$$

which is a Vandermonde determinant in the  $y_i$ s and  $x_i$ s times

$$\prod_{1 \leq i < j \leq n_c} (x_i - x_j) \prod_{1 \leq i < j \leq n_d} (y_i - y_j).$$

The formula for these tau functions is no longer a sum of  $c_{n_c, n_d, n_e}$ s, since  $n_e$  is zero and, when  $k \geq \ell$ , there is only one choice for  $n_c$  and  $n_d$  such that  $n_c + n_d = k$  and  $n_d = \ell$ . When  $k < \ell$ , no choice of  $n_c$  and  $n_d$  exists. We then have, when  $k \geq \ell$ ,

$$\tau_{k,\ell}^{(\alpha, \beta)} = \frac{1}{(k-\ell)! \ell!} \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{y}} \left( \prod_{i=1}^{k-\ell} C^{(\alpha-\beta)}(x_i) \prod_{i=1}^{\ell} D^{(\alpha)}(y_i) p_{k-\ell, \ell} \right).$$

Since we have no  $e_i$ s in our formulas, the  $S_e^\pm(z)$  act trivially on our tau functions, so we only need  $S_c^\pm(z)$  and  $S_d^\pm(z)$ . So we have  $z^k S_c^+(z) S_d^+(z) \tau_{k,\ell}^{(\alpha, \beta)} =$

$$(-1)^{\frac{\ell(\ell+1)}{2}} \det \begin{bmatrix} d_\alpha & \cdots & d_{\alpha+\ell-1} & c_{\alpha-\beta} & \cdots & c_{\alpha-\beta+k-\ell-1} & 1 \\ d_{\alpha+1} & \cdots & d_{\alpha+\ell} & c_{\alpha-\beta+1} & \cdots & c_{\alpha-\beta+k-\ell} & z \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ d_{\alpha+k} & \cdots & d_{\alpha+k-\ell-1} & c_{\alpha-\beta+k} & \cdots & c_{\alpha-\beta+2k-\ell-1} & z^k \end{bmatrix},$$

which comes from the fact that

$$S_c^+(z) S_d^+(z) \tau_{k,\ell}^{(\alpha, \beta)} = \frac{1}{(k-\ell)! \ell!} \text{Res}_{\mathbf{x}} \text{Res}_{\mathbf{y}} \left( \prod_{i=1}^{k-\ell} C^{(\alpha-\beta)}(x_i) \prod_{i=1}^{\ell} D^{(\alpha)}(y_i) \prod_{i=1}^{k-\ell} (z - x_i) \prod_{i=1}^{\ell} (z - y_i) p_{k-\ell, \ell} \right)$$

and

$$\prod_{i=1}^{k-\ell} (z - x_i) \prod_{i=1}^{\ell} (z - y_i) p_{k-\ell, \ell}$$

is a Vandermonde determinant in the  $y_i$ s,  $x_i$ s, and  $z$  times

$$\prod_{1 \leq i < j \leq k-\ell} (x_i - x_j) \prod_{1 \leq i < j \leq \ell} (y_i - y_j).$$

Denote  $p_{k, \ell}^{(\alpha, \beta)}(z) = \frac{1}{\tau_{k, \ell}^{(\alpha, \beta)}} z^k S_c^+(z) S_d^+(z) \tau_{k, \ell}^{(\alpha, \beta)}$ , which is a monic polynomial of degree  $k$ . Given  $\alpha, \beta \in \mathbb{Z}$ , define  $\langle \cdot, \cdot \rangle_C$  to be the bilinear product given by

$$\langle f(z), g(z) \rangle_C = \text{Res}_z \left( \sum_{i=0}^{\infty} \frac{c_{\alpha-\beta+i}}{z^{i+1}} f(z) g(z) \right) = 0,$$

for all polynomials  $f(z)$  and  $g(z)$ . Similarly, define  $\langle \cdot, \cdot \rangle_D$  to be the bilinear product given by

$$\langle f(z), g(z) \rangle_D = \text{Res}_z \left( \sum_{i=0}^{\infty} \frac{d_{\alpha+i}}{z^{i+1}} f(z) g(z) \right) = 0.$$

Consider the second row and first column of

$$(\Psi^{[0,0]}(\alpha, \beta))^{-1} \Psi^{[k, \ell]}(\alpha, \beta) = \begin{bmatrix} 1 & 0 & 0 \\ -\sum_{i=0}^{\infty} \frac{c_{\alpha-\beta+i}}{z^{i+1}} & 1 & 0 \\ -\sum_{i=0}^{\infty} \frac{d_{\alpha+i}}{z^{i+1}} & 0 & 1 \end{bmatrix} \times \frac{1}{\tau_{k, \ell}^{(\alpha, \beta)}} \begin{bmatrix} z^k S_c^+(z) S_d^+(z) \tau_{k, \ell}^{(\alpha, \beta)} & z^k \frac{S_c^+(z) S_d^+(z) \tau_{k-1, \ell}^{(\alpha, \beta)}}{z} & (-1)^k z^k \frac{S_c^+(z) S_d^+(z) \tau_{k-1, \ell-1}^{(\alpha, \beta)}}{z} \\ z^{\ell-k} \frac{S_c^-(z) \tau_{k+1, \ell}^{(\alpha, \beta)}}{z} & z^{\ell-k} S_c^-(z) \tau_{k, \ell}^{(\alpha, \beta)} & z^{\ell-k} \frac{S_c^-(z) \tau_{k, \ell-1}^{(\alpha, \beta)}}{z} \\ (-1)^{k+1} z^{-\ell} \frac{S_d^-(z) \tau_{k+1, \ell+1}^{(\alpha, \beta)}}{z} & (-1)^k z^{-\ell} \frac{S_d^-(z) \tau_{k, \ell+1}^{(\alpha, \beta)}}{z} & z^{-\ell} S_d^-(z) \tau_{k, \ell}^{(\alpha, \beta)} \end{bmatrix}.$$

Since  $z^{\ell-k} \frac{S_c^-(z) \tau_{k+1, \ell}^{(\alpha, \beta)}}{z}$  has highest degree  $\ell - k - 1$ , we see that

$$\langle p_{k, \ell}^{(\alpha, \beta)}, z^n \rangle_C = 0$$

for  $0 \leq n < k - \ell$ . Similarly, the entry in the third row and first column gives

$$\langle p_{k, \ell}^{(\alpha, \beta)}, z^n \rangle_D = 0$$

for  $0 \leq n < \ell$ , since  $(-1)^{k+1} z^{-\ell} \frac{S_d^-(z) \tau_{k+1, \ell+1}^{(\alpha, \beta)}}{z}$  has degree at most  $-\ell - 1$ . Such polynomials,  $p_{k, \ell}^{(\alpha, \beta)}(z)$ , are known as “type II multiple orthogonal polynomials” (see [7], [16]).

We plan to continue studying orthogonal polynomials coming from the  $\widehat{GL}_3$  case. In particular, we would like to better understand more general cases, in which the series  $E^{(\beta)}(z)$  is not required to be 0.

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